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Complex multi-projective variety and entanglement

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Abstract
In this paper, we will show that a vanishing generalized concurrence of a separable state can be seen as an algebraic variety called the Segre variety. This variety defines a quadric space which gives a geometric picture of separable states. For pure, bi- and three-partite states the variety equals the generalized concurrence. Moreover, we generalize the Segre variety to a general multipartite state by relating to a quadric space defined by two-by-two subdeterminants.

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1. Introduction

The most interesting feature of quantum mechanical systems, namely, quantum entanglement, was defined by Schrödinger [1] and Einstein, Podolsky and Rosen [2]. Many years have passed since the dawn of quantum mechanics, but we have still not been able to solve the enigma of entanglement, e.g., finding a complete mathematical model to describe, quantify, and at the same time reveal the physical implications of this feature. Moreover, we know very little about the geometry of entanglement. In quantum mechanics, the space of a pure state can be described by the N-dimensional complex projective space $\mathbf{CP}^N$. The question now is, how can we define quantum entanglement of a general pure state on such complex projective space?

There are several different answers to this question. One of the earliest proposals was to quantify the entanglement in terms of a distance to the nearest separable state [3]. Another idea is to use the maximum violation of generalized Bell inequalities as a measure of entanglement [4]. Such Bell inequality functions are called entanglement witnesses, and have mostly been used to detect nonseparable states [5–7]. However, in a recent paper, Bertlmann, Narnhofer and Thirring have combined the two ideas and shown that the maximal violation of a generalized Bell inequalities and the Hilbert–Schmidt distance to the convex set of separable states are equivalent [8]. Hence, they demonstrate that both these concepts have a geometric interpretation. Yet another idea to quantify entanglement is to use the entropy of the reduced density matrix as a measure of entanglement, the so-called entanglement of formation [9]. If the entropy of the remaining subsystem is the same as that for the original system, there is no entanglement between the remaining subsystem and the subsystem being traced out. For bipartite, pure states, the entanglement of formation is simply a entropic function of the state's so-called concurrence [10]. In this paper, we shall demonstrate that concurrence, just like entanglement witnesses, has a geometric interpretation. The connection between concurrence and geometry is found in a map called a Segre embedding, see Brody and Hughston [11]. They illustrate this map for a pair of qubits, and point out that this map characterizes the idea of quantum entanglement. Moreover, they define a variety that represents the set of separable states but they do not discuss it much further. Segre embedding has also been discussed by Miyake [12] in the context of classification of multipartite states in entanglement classes (where two states belong to the same class if they are interconvertible under stochastic local operations and classical communication).

In this paper, we will expand this idea and describe the Segre variety, which is a quadric space in algebraic geometry, by giving a complete and explicit formula for it. Moreover, we will compare the Segre variety with the concurrence of a general pure, bipartite state [13–17]. Vanishing of the concurrence of a separable state coincides with the Segre variety. This will illustrate the geometry of concurrence as a measure of bipartite entanglement in a complete and satisfactory way. Furthermore, we generalize Segre variety to a general multipartite state by relating the decomposable tensors to a quadric space defined by two-by-two prime ideals. In this paper, we assume that the reader is familiar with basic concepts in abstract algebra such as ring theory and fields.

2. Quantum entanglement

In this section, we will define separable states and entangled states. Let us denote a general, pure, composite quantum system with m subsystems, $\mathcal{Q} = \mathcal{Q}_m^p(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of a state

$$|\Psi\rangle = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\cdots\sum_{i_m=1}^{N_m}\alpha_{i_1,i_2,\ldots,i_m}|i_1, i_2, \ldots, i_m\rangle \tag{1}$$

defined on a Hilbert space

$$\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m} = \mathbf{C}^{N_1}\otimes\mathbf{C}^{N_2}\otimes\cdots\otimes\mathbf{C}^{N_m}, \tag{2}$$

where the dimension of the jth Hilbert space is given by $N_j = \dim\left(\mathcal{H}_{\mathcal{Q}_j}\right)$. We are going to use this notation throughout this paper, i.e., we denote a pure pair of qubits by $\mathcal{Q}_2^p(2,2)$. Next, let $\rho_{\mathcal{Q}}$ denote a density operator acting on $\mathcal{H}_{\mathcal{Q}}$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{\mathrm{sep}}$, with respect to the Hilbert space decomposition, if it can be written as

$$\rho_{\mathcal{Q}}^{\mathrm{sep}} = \sum_{k=1}^{\mathrm{N}} p_k \bigotimes_{j=1}^{m}\rho_{\mathcal{Q}_j}^{k}, \qquad \sum_{k=1}^{N} p_k = 1 \tag{3}$$

for some positive integer N, where p_k are positive real numbers and $\rho_{\mathcal{Q}_j}^{k}$ denote a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^{p}$ represents a pure state, then the quantum system is fully separable if $\rho_{\mathcal{Q}}^{p}$ can be written as $\rho_{\mathcal{Q}}^{\mathrm{sep}} = \bigotimes_{j=1}^{m}\rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is a density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is called an entangled state. Some of the generic entangled states are called Bell states and EPR states.

3. Segre variety

This section serves as an introduction to the affine space, Segre embedding and the Segre variety in such a way that it enables us to establish a relation between concurrence and Segre variety in the following sections. The general references for this section are [18–22]. Let $\mathbf{C}$ be a field of complex numbers and N be an integer. Then, we define an N-dimensional affine space over $\mathbf{C}$, denoted by $\mathcal{A}^N_{\mathbf{C}}$ or $\mathcal{A}^N$, to be the set of all N-tuples of elements of $\mathbf{C}$, i.e.,

$$\mathcal{A}^N = \{P = (a_1, a_2, \ldots, a_N) : a_1, a_2, \ldots, a_N \in \mathbf{C}\}. \tag{4}$$

An element $P = (a_1, a_2, \ldots, a_N)$ is called a point, where $a_i \in \mathbf{C}$ is called a coordinate of P. In general, we call $\mathcal{A}^1 = \mathbf{C}$ the affine line and $\mathcal{A}^2$ the affine plane.

Let $R(N) = \mathbf{C}[Z_1, Z_1, \ldots, Z_N]$ be the polynomial ring over $\mathbf{C}$ in the N variables $Z_1, Z_1, \ldots, Z_N$. Any element $F \in R(N)$ gives rise to a $\mathbf{C}$-valued map on $\mathcal{A}^N$ by evaluation, i.e., $P = (a_1, a_2, \ldots, a_N) \longmapsto F(a_1, a_2, \ldots, a_N) = F(P)$. Such a function on $\mathcal{A}^N$ is called a polynomial or a regular function. Given $F \in R(n)$, the set of points yielding zeros of F is denoted by $\mathcal{V}(F)$, i.e.,

$$\mathcal{V}(F) = \{P \in \mathcal{A}^N : F(P) = 0\}. \tag{5}$$

A closed subset of $\mathcal{A}^N$ which is of the form $\mathcal{V}(F)$, with $F \in R(N)$ not a scalar, is called the hypersurface defined by F or the hypersurface whose equation is $F = 0$. If $F \in R(N)$ is of degree $r \geqslant 1$, then $\mathcal{V}(F)$ is called a hypersurface of degree r in $\mathcal{A}^N$. It is called a hyperplane, a quadric, a cubic, $\ldots$, for $r = 1, 2, 3, \ldots$. The union of a finite number of hypersurfaces is again a hypersurface and its degree is the sum of their degrees, i.e.,

$$\mathcal{V}(F_1 F_2 \cdots F_d) = \mathcal{V}\left(\bigcap_{i=1}^{r} F_i\right) = \mathcal{V}(F_1) \cup \mathcal{V}(F_2) \cup \cdots \cup \mathcal{V}(F_1). \tag{6}$$

A subset $\mathcal{I}$ of a commutative ring R is called an ideal of R if it has the following properties: (i) for any elements $\alpha, \beta \in \mathcal{I}$, we have $\alpha + \beta \in \mathcal{I}$. (ii) For any elements $a \in R$ and $\alpha \in \mathcal{I}$, we have $a\alpha \in \mathcal{I}$. If two elements $a \neq 0, b \neq 0$ of R satisfy $ab = 0$, then we call a a zero divisor (and so b). R is called an integral domain if it has no zero divisor and an ideal $\mathcal{I}$ of R is called a prime ideal if $R/\mathcal{I}$ is an integral domain. The ideal $\mathcal{I}(V)$ of an algebraic subset $V \subset \mathcal{A}^N$ is the largest ideal of polynomial functions on $\mathcal{A}^N$ vanishing on V and the coordinate ring $\mathbf{C}[V]$ of V is naturally isomorphic to quotient ring $R(N)/\mathcal{I}(V)$. $\mathbf{C}[V]$ is reduced and V is said to be equipped with the canonical reduced structure. An irreducible algebraic subset V of $\mathcal{A}^N$ is called an affine algebraic variety, i.e., if its ideal $\mathcal{I}(V)$ is a prime ideal of $R(N)$ or equivalently, its coordinate ring $\mathbf{C}[V] = R(N)/\mathcal{I}(V)$ is an integral domain.

Now, let $\mathcal{A}^{N_1}$ and $\mathcal{A}^{N_2}$ be affine spaces. If $X = (x_1, x_2, \ldots, x_{N_1})$ and $Y = \left(y_1, y_2, \ldots, y_{N_2}\right)$ are two points defined on $\mathcal{A}^{N_1}$ and $\mathcal{A}^{N_2}$, respectively, then the map

$$\begin{aligned} \phi : \mathcal{A}^{N_1} \times \mathcal{A}^{N_2} \quad &\longrightarrow \quad \mathcal{A}^{N_1+N_2} \\ (X, Y) \quad &\longmapsto \quad \left(x_1, x_2, \ldots, x_{N_1}, y_1, y_2, \ldots, y_{N_2}\right) \end{aligned} \tag{7}$$

is a one-to-one and onto mapping. If $\mathcal{V}$ and $\mathcal{U}$ are algebraic sets in $\mathcal{A}^{N_1}$ and $\mathcal{A}^{N_2}$, respectively, then $\phi(\mathcal{V} \times \mathcal{U})$ is a algebraic set in $\mathcal{A}^{N_1+N_2}$.

If $X = (x_1, x_2, \ldots, x_N)$ and $Y = (y_1, y_2, \ldots, y_N)$ are two different points in $\mathcal{A}^N$, then the line $\mathcal{L}$ passing through X and Y is parametrically defined as

$$\mathcal{L} = \{(\delta x_1 + \tau y_1, \delta x_2 + \tau y_2, \ldots, \delta x_N + \tau y_N) : \delta, \tau \in \mathbf{C}\}. \tag{8}$$

The complex projective space, $\mathbf{CP}^{N-1}$, is defined as the set of all lines through $(0, 0, \ldots, 0)$ in $\mathcal{A}^N$. Let X and Y be two points. Then X and Y determines the same line if, and only if, there exist a $\delta \in \mathbf{C}$, $\delta \neq 0$, such that $y_i = \delta x_i$, for all $i = 1, 2, \ldots, N$. That is, the lines X and Y are equivalent, which we denote by $X \sim Y$. Now, if we assume that this is the case, then

$$\mathbf{CP}^{N-1} \cong \frac{\mathcal{A}^N - \{(0, 0, \ldots, 0)\}}{X \sim \delta X}. \tag{9}$$

If a point $X \in \mathbf{CP}^{N-1}$ is determined by $(x_1, x_2, \ldots, x_N) \in \mathcal{A}^N$, then we say that $(x_1, x_2, \ldots, x_N)$ is a set of homogeneous coordinates for X. If $x_i \neq 0$, then we have

$$X = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}\right). \tag{10}$$

Let $R = R(N) = \mathbf{C}[Z_0, Z_1, \ldots, Z_N]$ be the polynomial ring over $\mathbf{C}$ in the variables $Z_0, Z_1, \ldots, Z_N$. Then, for a form $F \in R$, we define $\mathcal{V}(F) = \{P \in \mathbf{CP}^{N-1} : F(P) = 0\}$, called the set of projective zeros of F. Unlike in the affine case, we have $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \neq \mathbf{CP}^{N_1+N_2-2}$. For example, in $\mathbf{CP}^1 \times \mathbf{CP}^1$, the lines $\mathcal{L}_x = \{x\} \times \mathbf{CP}^1$ and $\mathcal{L}_y = \{y\} \times \mathbf{CP}^1$ are parallel for $x \neq y$ in $\mathbf{CP}^1$ but there are no parallel lines in $\mathbf{CP}^2$ since any two distinct lines $L_1 = \mathcal{V}(a_1X_1 + a_2X_2 + a_3X_3)$ and $L_2 = \mathcal{V}(b_1X_1 + b_2X_2 + b_3X_3)$ intersect at the unique point $(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

Now, we want to make $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1}$ into a projective variety by its Segre embedding which we construct as follows: let X and Y be two points defined on $\mathbf{CP}^{N_1-1}$ and $\mathbf{CP}^{N_2-1}$, respectively. Then, the map

$$\begin{array}{rcl} \mathcal{S}_{N_1,N_2} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} & \longrightarrow & \mathbf{CP}^{N_1N_2-1} \\ (X, Y) & \longmapsto & \left(x_1y_1, \ldots, x_1y_{N_2}, \ldots, x_{N_1}y_1, \ldots, x_{N_1}y_{N_2}\right) \end{array} \tag{11}$$

is a closed immersion, called the Segre embedding. To see that, let X_i, and Y_j be the homogeneous coordinate functions on $\mathbf{CP}^{N_1-1}$ and $\mathbf{CP}^{N_2-1}$, respectively. Moreover, let $Z_{i,j}$ be the homogeneous coordinate function on $\mathbf{CP}^{N_1N_2-1}$. Now, we arrange the homogeneous coordinate $Z_{i,j}$ as follows:

$$\begin{pmatrix} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,N_2} \\ Z_{2,1} & Z_{2,2} & \cdots & Z_{2,N_2} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N_1,1} & Z_{N_1,2} & \cdots & Z_{N_1,N_2} \end{pmatrix}. \tag{12}$$

The map $\mathcal{S}_{N_1,N_2} = (\ldots, X_iY_j, \ldots)$ is a morphism since it is defined by polynomials on any affine piece $U_i \times U_j$ where

$$\mathbf{CP}^{N_1-1} = \bigcup_{i=1}^{N_1-1} U_i \quad \text{and} \quad \mathbf{CP}^{N_2-1} = \bigcup_{j=1}^{N_2-1} U_j \tag{13}$$

are the standard affine coverings. But the determinant

$$\det\begin{pmatrix} X_iY_k & X_iY_l \\ X_jY_k & X_jY_l \end{pmatrix} \tag{14}$$

vanishes for all i, j and k, l, so the image of $\mathcal{S}_{N_1,N_2}$ is contained in the closed subset

$$T = \left\{ (\ldots, z_{i,j}, \ldots) \in \mathbf{CP}^{N_1N_2-1} : \mathrm{rk}\begin{pmatrix} z_{1,1} & z_{1,2} & \cdots & z_{1,N_2} \\ z_{2,1} & z_{2,2} & \cdots & z_{2,N_2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{N_1,1} & z_{N_1,2} & \cdots & z_{N_1,N_2} \end{pmatrix} = 1 \right\},$$

where rk denotes the matrix rank. If Im denotes the image, then $T = \mathrm{Im}\left(\mathcal{S}_{N_1,N_2}\right)$ and $\mathcal{S}_{N_1,N_2}$ is an isomorphism. To see that, let us consider $t = (\ldots, z_{i,j}, \ldots) \in Z$. Then all the rows and columns of the rank one matrix $(z_{i,j})$ are proportional. For any columns $x \neq 0$ and any rows $y \neq 0$ of this matrix we have $t = \mathcal{S}_{N_1,N_2}(x, y)$ and $T = \mathrm{Im}\big(\mathcal{S}_{N_1,N_2}\big)$. Moreover, the map $t \longmapsto (x, y)$ is the inverse to $\mathcal{S}_{N_1,N_2}$ and so it is an isomorphism. If $V \subseteq \mathbf{CP}^{N_1-1}$ and $W \subseteq \mathbf{CP}^{N_2-1}$ are projective algebraic sets, then $V \times W$ is projective and is closed in the closed subvariety $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} = \mathrm{Im}\big(\mathcal{S}_{N_1,N_2}\big) \subset \mathbf{CP}^{N_1N_2-1}$. The image of the Segre embedding is an intersection of a family of quadric hypersurfaces in $\mathbf{CP}^{N_1N_2-1}$, that is

$$\mathrm{Im}\big(\mathcal{S}_{N_1,N_2}\big) = \bigcap_{i,j,k,l} \mathcal{V}(Z_{i,k}Z_{j,l} - Z_{i,l}Z_{j,k}). \tag{15}$$

i.e., $\mathrm{Im}(\mathcal{S}_{2,2}) = \mathcal{V}(Z_{1,1}Z_{2,2} - Z_{1,2}Z_{2,1})$ is a quadric surface in $\mathbf{CP}^3$.

3.1. Segre variety for a general bipartite state and concurrence

For given quantum system $\mathcal{Q}_2(N_1, N_2)$ we want make $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1}$ into a projective variety by its Segre embedding which we construct as follows. Let $\big(\alpha_1, \alpha_2, \ldots, \alpha_{N_1}\big)$ and $\big(\alpha_1, \alpha_2, \ldots, \alpha_{N_2}\big)$ be two points defined on $\mathbf{CP}^{N_1-1}$ and $\mathbf{CP}^{N_2-1}$, respectively, then the Segre map

$$\mathcal{S}_{N_1,N_2} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \longrightarrow \mathbf{CP}^{N_1N_2-1} \tag{16}$$

$$\big(\big(\alpha_1, \alpha_2, \ldots, \alpha_{N_1}\big), \big(\alpha_1, \alpha_2, \ldots, \alpha_{N_2}\big)\big) \longmapsto \big(\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,N_1}, \ldots, \alpha_{N_1,1}, \ldots, \alpha_{N_1,N_2}\big) \tag{17}$$

is well defined. Next, let $\alpha_{i,j}$ be the homogeneous coordinate function on $\mathbf{CP}^{N_1N_2-1}$. Then the image of the Segre embedding is an intersection of a family of quadric hypersurfaces in $\mathbf{CP}^{N_1N_2-1}$, that is

$$\mathrm{Im}\big(\mathcal{S}_{N_1,N_2}\big) = \bigcap_{i,j,k,l} \mathcal{V}(\mathcal{C}_{i,j;k,l}(N_1, N_2)) = \bigcap_{i,j,k,l} \mathcal{V}(\alpha_{i,k}\alpha_{j,l} - \alpha_{i,l}\alpha_{j,k}). \tag{18}$$

This quadric space is the space of separable states and it coincides with the definition of general concurrence $\mathcal{C}(\mathcal{Q}_2(N_1, N_2))$ of a pure bipartite state [13, 14] because

$$\begin{aligned} \mathcal{C}(\mathcal{Q}_2(N_1, N_2)) &= \left(\mathcal{N} \sum_{j,i=1}^{N_1} \sum_{l,k=1}^{N_2} |\mathcal{C}_{i,j;k,l}(N_1, N_2)|^2\right)^{\frac{1}{2}} \\ &= \left(\mathcal{N} \sum_{j,i=1}^{N_1} \sum_{l,k=1}^{N_2} |\alpha_{i,k}\alpha_{j,l} - \alpha_{i,l}\alpha_{j,k}|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{19}$$

where $\mathcal{N}$ is a somewhat arbitrary normalization constant. The separable set is defined by $\alpha_{i,k}\alpha_{j,l} = \alpha_{il}\alpha_{jk}$ for all i, j and k, l, i.e.,

$$\mathrm{Im}(\mathcal{S}_{2,2}) = \mathcal{V}(\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}) \quad \Longleftrightarrow \quad \alpha_{1,1}\alpha_{2,2} = \alpha_{1,2}\alpha_{2,1} \tag{20}$$

is a quadric surface in $\mathbf{CP}^3$ which coincides with the space of separable set of pairs of qubits.

4. Multi-projective variety and multi-partite entanglement measure

In this section, we will generalize the Segre variety to a multi-projective space. As in the previous section, we can make $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1}$ into a projective variety by its Segre embedding following almost the same procedure. Let $(\alpha_1, \alpha_2, \dots, \alpha_{N_i})$ be points defined on $\mathbf{CP}^{N_i-1}$. Then the Segre map

$$\begin{array}{rcl} \mathcal{S}_{N_1,\dots,N_m} : \mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \cdots \times \mathbf{CP}^{N_m-1} & \longrightarrow & \mathbf{CP}^{N_1 N_2 \cdots N_m - 1} \\ ((\alpha_1, \alpha_2, \dots, \alpha_{N_1}), \dots, (\alpha_1, \alpha_2, \dots, \alpha_{N_m})) & \longmapsto & (\dots, \alpha_{i_1,i_2,\dots,i_m}, \dots) \end{array} \tag{21}$$

is well defined for $\alpha_{i_1 i_2 \cdots i_m}$, $1 \leqslant i_1 \leqslant N_1$, $1 \leqslant i_2 \leqslant N_2, \dots, 1 \leqslant i_m \leqslant N_m$ as a homogeneous coordinate function on $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$. Now, let us consider the composite quantum system $\mathcal{Q}_m^p(N_1, N_2, \dots, N_m)$ and let the coefficients of $|\Psi\rangle$, namely $\alpha_{i_1,i_2,\dots,i_m}$, make an array as follows:

$$\mathcal{A} = (\alpha_{i_1,i_2,\dots,i_m})_{1 \leqslant i_j \leqslant N_j}, \tag{22}$$

for all $j = 1, 2, \dots, m$. $\mathcal{A}$ can be realized as the following set $\{(i_1, i_2, \dots, i_m) : 1 \leqslant i_j \leqslant N_j, \forall j\}$, in which each point $(i_1, i_2, \dots, i_m)$ is assigned the value $\alpha_{i_1,i_2,\dots,i_m}$. Then $\mathcal{A}$ and its realization is called an m-dimensional box-shape matrix of size $N_1 \times N_2 \times \cdots \times N_m$, where we associate with each such matrix a sub-ring $S_{\mathcal{A}} = \mathbf{C}[\mathcal{A}] \subset S$, where S is a commutative ring over the complex number field. For each $j = 1, 2, \dots, m$, a two-by-two minor about the jth coordinate of $\mathcal{A}$ is given by

$$\mathcal{C}_{k_1,l_1;k_2,l_2;\dots;k_m,l_m} = \alpha_{k_1,k_2,\dots,k_m} \alpha_{l_1,l_2,\dots,l_m} - \alpha_{k_1,k_2,\dots,k_{j-1},l_j,k_{j+1},\dots,k_m} \alpha_{l_1,l_2,\dots,l_{j-1},k_j,l_{j+1},\dots,l_m} \in S_{\mathcal{A}}. \tag{23}$$

Then the ideal $\mathcal{I}_{\mathcal{A}}^m$ of $\mathrm{S}_{\mathcal{A}}$ is generated by $\mathcal{C}_{k_1,l_1;k_2,l_2;\dots;k_m,l_m}$ and describes the separable states in $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$ [23]. The image of the Segre embedding $\mathrm{Im}(\mathcal{S}_{N_1,N_2,\dots,N_m})$ which again is an intersection of families of quadric hypersurfaces in $\mathbf{CP}^{N_1 N_2 \cdots N_m - 1}$ is given by

$$\mathrm{Im}(\mathcal{S}_{N_1,N_2,\dots,N_m}) = \bigcap_{\forall j} \mathcal{I}_{\mathcal{A}}^m = \bigcap_{\forall j} \mathcal{V}\left(\mathcal{C}_{k_1,l_1;k_2,l_2;\dots;k_m,l_m}\right). \tag{24}$$

Moreover, following the same argumentation as in the bipartite case, we can define an entanglement measure for a pure multipartite state as

$$\begin{aligned} \mathcal{E}(\mathcal{Q}_m^p(N_1, \dots, N_m)) &= \left(\mathcal{N} \sum_{\forall j} \left| \mathcal{C}_{k_1,l_1;k_2,l_2;\dots;k_m,l_m} \right|^2 \right)^{\frac{1}{2}} \\ &= \left(\mathcal{N} \sum_{\forall j} \left| \alpha_{k_1,k_2,\dots,k_m} \alpha_{l_1,l_2,\dots,l_m} - \alpha_{k_1,k_2,\dots,k_{j-1},l_j,k_{j+1},\dots,k_m} \alpha_{l_1,l_2,\dots,l_{j-1},k_j,l_{j+1},\dots,l_m} \right|^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{25}$$

where $\mathcal{N}$ is an arbitrary normalization constant and $j = 1, 2, \dots, m$. This measure coincides with the concurrence for a general bipartite and three-partite state. However, for reasons that will be explained below, it fails to quantify the entanglement for $m \geqslant 4$, whereas it still provides the condition of full separability.

5. Example: three-partite state

As an example, let us look a general three-partite state. The generalized concurrence [13] for such a state is given by

$$\mathcal{E}\big(\mathcal{Q}_3^p(N_1, N_2, N_3)\big) = \left(\mathcal{N} \sum_{k_1,l_1;k_2,l_2;k_3,l_3} \sum_{\forall j} \left|\mathcal{C}_{k_1,l_1;k_2,l_2;k_3,l_3}\right|^2\right)^{\frac{1}{2}}$$
$$= \left(\mathcal{N} \sum_{k_1,l_1;k_2,l_2;k_3,l_3} \big(\left|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3} - \alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3}\right|^2 + \left|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3} - \alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3}\right|^2\big) + \left|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3} - \alpha_{l_1,k_2,k_3}\alpha_{k_1,l_2,l_3}\right|^2\right)^{\frac{1}{2}}. \tag{26}$$

This equation for an entanglement measure is equivalent but not equal to our entanglement tensor based on joint POVMs on phase space [24]. For a three-qubit state $\mathcal{Q}_3^p(2, 2, 2)$, we have

$$\begin{aligned}\mathcal{E}\big(\mathcal{Q}_3^p(2, 2, 2)\big) = (4\mathcal{N}\{&2|\alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1}|^2 + 2|\alpha_{1,1,2}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,2}|^2\\
&+2|\alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}|^2 + 2|\alpha_{1,2,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,2,1}|^2\\
&+2|\alpha_{1,1,1}\alpha_{1,2,2} - \alpha_{1,1,2}\alpha_{1,2,1}|^2 + 2|\alpha_{2,1,1}\alpha_{2,2,2} - \alpha_{2,1,2}\alpha_{2,2,1}|^2\\
&+|\alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,1,2}\alpha_{2,2,1}|^2 + |\alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,1}\alpha_{2,1,2}|^2\\
&+|\alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,1}|^2 + |\alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,2}|^2\\
&+|\alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,2}\alpha_{2,1,1}|^2 + |\alpha_{1,2,1}\alpha_{2,1,2} - \alpha_{1,2,2}\alpha_{2,1,1}|^2\})^{\frac{1}{2}}.\end{aligned} \tag{27}$$

We can derive this expression in a different way than it was originally derived using the idea of the Segre ideal. The ideal $\mathcal{I}^{2,2,2}_{\mathcal{Q}_1\models\mathcal{Q}_2\mathcal{Q}_3}$ representing if a subsystem $\mathcal{Q}_1$ that is unentangled with $\mathcal{Q}_2\mathcal{Q}_3$ is generated by the six two-by-two subdeterminants of

$$\begin{pmatrix}\alpha_{1,1,1} & \alpha_{1,1,2} & \alpha_{1,2,1} & \alpha_{1,2,2}\\ \alpha_{2,1,1} & \alpha_{2,1,2} & \alpha_{2,2,1} & \alpha_{2,2,2}\end{pmatrix} \tag{28}$$

and is given by

$$\begin{aligned}\mathcal{I}^{2,2,2}_{\mathcal{Q}_1\models\mathcal{Q}_2\mathcal{Q}_3} = \langle&\alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}, \alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1}, \alpha_{1,1,1}\alpha_{2,2,2}\\
&- \alpha_{1,2,2}\alpha_{2,1,1}, \alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,2}, \alpha_{1,1,2}\alpha_{2,2,2}\\
&- \alpha_{1,2,2}\alpha_{2,1,2}, \alpha_{1,2,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,2,1}\rangle,\end{aligned}$$

where we have used the notation $\models$ to indicate that $\mathcal{Q}_1$ is separated from $\mathcal{Q}_2\mathcal{Q}_3$ but we still could have entanglement between $\mathcal{Q}_2$ and $\mathcal{Q}_3$. The notation $\{2, 2, 2\}$ is used to indicate a three-partite state where the dimension of the Hilbert space of each subsystem is 2 (i.e., three qubits). In the same way, we can define the ideal $\mathcal{I}^{2,2,2}_{\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_3}$ representing if the subsystem $\mathcal{Q}_2$ is unentangled with $\mathcal{Q}_1\mathcal{Q}_3$ and $\mathcal{I}_{\mathcal{Q}_3\models\mathcal{Q}_1\mathcal{Q}_2}$ representing if the subsystem $\mathcal{Q}_3$ is unentangled with $\mathcal{Q}_2\mathcal{Q}_3$. The ideals are generated by the six two-by-two subdeterminants of

$$\begin{pmatrix}\alpha_{1,1,1} & \alpha_{1,1,2} & \alpha_{2,1,1} & \alpha_{2,1,2}\\ \alpha_{1,2,1} & \alpha_{1,2,2} & \alpha_{2,2,1} & \alpha_{2,2,2}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\alpha_{1,1,1} & \alpha_{1,2,1} & \alpha_{2,1,1} & \alpha_{2,2,1}\\ \alpha_{1,1,2} & \alpha_{1,2,2} & \alpha_{2,1,2} & \alpha_{2,2,2}\end{pmatrix}, \tag{29}$$

respectively. Written out explicitly they are

$$\begin{aligned}\mathcal{I}^{2,2,2}_{\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_3} = \langle&\alpha_{1,1,1}\alpha_{1,2,2} - \alpha_{1,1,2}\alpha_{1,2,1}, \alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{2,1,1}\alpha_{1,2,1}, \alpha_{1,1,1}\alpha_{2,2,2}\\
&- \alpha_{2,1,2}\alpha_{1,2,1}, \alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{2,1,1}\alpha_{1,2,2}, \alpha_{1,1,2}\alpha_{2,2,2}\\
&- \alpha_{1,2,2}\alpha_{2,1,2}, \alpha_{2,1,1}\alpha_{2,2,2} - \alpha_{2,1,2}\alpha_{2,2,1}\rangle,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}^{2,2,2}_{\mathcal{Q}_3\models\mathcal{Q}_1\mathcal{Q}_2} = \langle&\alpha_{1,1,1}\alpha_{1,2,2} - \alpha_{1,2,1}\alpha_{1,1,2}, \alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{2,1,1}\alpha_{1,1,2}, \alpha_{1,1,1}\alpha_{2,2,2}\\
&- \alpha_{2,2,1}\alpha_{1,1,2}, \alpha_{1,2,1}\alpha_{2,1,2} - \alpha_{2,1,1}\alpha_{1,2,2}, \alpha_{1,2,1}\alpha_{2,2,2}\\
&- \alpha_{2,2,1}\alpha_{1,2,2}, \alpha_{2,1,1}\alpha_{2,2,2} - \alpha_{2,2,1}\alpha_{2,1,2}\rangle.\end{aligned}$$

Hence, the Segre ideal of a completely separable pure three-qubit state is given by

$$
\begin{aligned}
\mathcal{I}^{2,2,2}_{\text{Segre}} &= \mathcal{I}^{2,2,2}_{\{\mathcal{Q}_1\models\mathcal{Q}_2\mathcal{Q}_3,\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_3,\mathcal{Q}_3\models\mathcal{Q}_1\mathcal{Q}_2\}} \\
&= \mathcal{I}^{2,2,2}_{\mathcal{Q}_1\models\mathcal{Q}_2\mathcal{Q}_3} \bigcap \mathcal{I}^{2,2,2}_{\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_3} \bigcap \mathcal{I}^{2,2,2}_{\mathcal{Q}_2\models\mathcal{Q}_1\mathcal{Q}_2} \\
&= \langle \alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}, \alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1}, \alpha_{1,1,1}\alpha_{2,2,2} \\
&\quad - \alpha_{1,2,2}\alpha_{2,1,1}, \alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,2}, \alpha_{1,1,2}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,2}, \alpha_{1,2,1}\alpha_{2,2,2} \\
&\quad - \alpha_{1,2,2}\alpha_{2,2,1}, \alpha_{1,1,1}\alpha_{1,2,2} - \alpha_{1,1,2}\alpha_{1,2,1}, \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,1}\alpha_{2,1,2}, \alpha_{1,1,2}\alpha_{2,2,1} \\
&\quad - \alpha_{1,2,2}\alpha_{2,1,1}, , \alpha_{2,1,1}\alpha_{2,2,2} - \alpha_{2,1,2}\alpha_{2,2,1}, \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,1,2}\alpha_{2,2,1}, \alpha_{1,2,1}\alpha_{2,1,2} \\
&\quad - \alpha_{1,2,2}\alpha_{2,1,1}\rangle.
\end{aligned} \tag{30}
$$

This equation coincide with equation (24) for a three-qubit state. For a general multipartite state, that is, for $m \geqslant 4$ this measure $\mathcal{E}\big(\mathcal{Q}^p_m(N_1, \ldots, N_m)\big)$ is not invariant under local operations. To show why this measure is not invariant under local operations, let us consider the quantum system $\mathcal{Q}^p_4(2, 2, 2, 2)$. In this case, we can have seven types of separability between different subsystems as follows: it maybe possible to factor $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, or $\mathcal{Q}_4$ from the composite system. To check this we need to make four different permutations of indices and this is exactly what the measure $\mathcal{E}\big(\mathcal{Q}^p_4(2, 2, 2, 2)\big)$ does. But there are other types of separability in this four-qubit state, namely if it is possible to factor out $\mathcal{Q}_1\mathcal{Q}_2$, $\mathcal{Q}_1\mathcal{Q}_3$, $\mathcal{Q}_1\mathcal{Q}_4$, $\mathcal{Q}_2\mathcal{Q}_3$, $\mathcal{Q}_2\mathcal{Q}_4$ or $\mathcal{Q}_3\mathcal{Q}_4$. These six possible factorizations can be reduced to three checks of separability since if we test for separability of, i.e., $\mathcal{Q}_1\mathcal{Q}_2$, we have simultaneously tested $\mathcal{Q}_3\mathcal{Q}_4$. For these types of separability, we do need to perform more than one simultaneous permutation of indices. The measure (25) does not check this type of separability which is needed in the general case [25].

6. Conclusion

In this paper, we have discussed a geometric picture of the separable set of states for a general pure bipartite state based on algebraic complex projective geometry. In particular, we have proved that complete separability for a general pure bipartite state can be seen as a Segre variety. Moreover, we have generalized this result to multipartite states, by defining a map called multi-projective Segre embedding. The image of this map defines a quadric space, namely the generalized Segre variety which we constructed by a prime ideal of two-by-two subdeterminants of a so-called decomposable tensor. We showed that the Segre variety define the completely separable states of a general multipartite state. Furthermore, based on this subdeterminant, we define an entanglement measure for general pure bipartite and three-partite states which coincide with generalized concurrence.

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